# Four loop results for the 2D $O(n)$ nonlinear $\sigma$ model with 0–loop and 1–loop Symanzik actions

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We present complete three loop results and preliminary four loop results for the 2D $O(n)$ nonlinear $\sigma$ model with 0–loop and 1–loop Symanzik improved actions. This calculation aims to test the improvement in the numerical precision that the combination of Symanzik actions and effective couplings can give in Monte Carlo simulations.

## 1. INTRODUCTION

Monte Carlo simulations on the lattice are affected by systematic errors due to the finiteness of the lattice spacing $a$. Symanzik proposed a method to reduce these effects on the physical scaling by following a perturbative procedure [1].

The integration of the beta function of the theory allows to express the results of a simulation in physical units by fixing the lattice scale $a$. The asymptotic scaling regime is attained when the lattice scale is well determined with the first universal terms in the perturbative expansion of the beta function. However this regime is barely achieved and an estimate of the non–universal corrections to asymptotic scaling is necessary.

The asymptotic scaling regime can be more easily accomplished, even for moderately large couplings, if an effective coupling [2] is used.

We plan to test the numerical precision obtained in Monte Carlo simulations by using a combination of Symanzik improved actions together with an effective coupling. We want to perform this test on the 2D $O(n)$ nonlinear $\sigma$ model. In this proceeding we present the perturbative calculation of the Renormalization Group Invariant (RGI) functions. We also report on a recent calculation of the weak coupling expansion of the energy up to four loops [3] which is necessary to express the beta function in terms of the effective coupling.

## 2. SYMANZIK IMPROVED ACTIONS

We consider the 0–loop and the 1–loop Symanzik improved actions:

$$
\begin{aligned}
S^{0-\mathrm{loop}} &= \frac{a^2}{g}\sum_x \left(\frac{2}{3}\vec{\phi}_x K_1 \vec{\phi}_x - \frac{1}{24}\vec{\phi}_x K_2 \vec{\phi}_x\right) \ , \\
S^{1-\mathrm{loop}} &= \frac{a^2}{g}\sum_x \Bigg[ \frac{1}{2}\vec{\phi}_x K_1 \vec{\phi}_x - a^2 c_5\, g \left(K_1 \vec{\phi}_x\right)^2 \\
&- a^2\left(c_6\, g - \frac{1}{24}\right)\sum_\mu \left(\partial^+_\mu \partial^-_\mu \vec{\phi}_x\right)^2 \\
&- a^2 c_7\, g \left(\vec{\phi}_x K_1 \vec{\phi}_x\right)^2 - a^2 c_8\, g \sum_\mu \left(\vec{\phi}_x \partial^+_\mu \partial^-_\mu \vec{\phi}_x\right)^2 \\
&- \frac{1}{16} a^2 c_9\, g \sum_{\mu\nu} \left(\left(\partial^+_\mu + \partial^-_\mu\right)\vec{\phi}_x \cdot \left(\partial^+_\nu + \partial^-_\nu\right)\vec{\phi}_x\right)^2 \Bigg] \ .
\end{aligned}
\tag{1}
$$

The $n$–component scalar field $\vec{\phi}_x$ is constrained by $\vec{\phi}_x^2 = 1$. The lattice operators in (1) are

$$
\begin{aligned}
K_1 \vec{\phi}_x &\equiv \frac{1}{a^2}\sum_\mu \left(2\vec{\phi}_x - \vec{\phi}_{x+\hat{\mu}} - \vec{\phi}_{x-\hat{\mu}}\right) \ , \\
K_2 \vec{\phi}_x &\equiv \frac{1}{a^2}\sum_\mu \left(2\vec{\phi}_x - \vec{\phi}_{x+2\,\hat{\mu}} - \vec{\phi}_{x-2\,\hat{\mu}}\right) \ , \\
\partial^+_\mu \vec{\phi}_x &\equiv \frac{1}{a}\left(\vec{\phi}_{x+\hat{\mu}} - \vec{\phi}_x\right) \ , \\
\partial^-_\mu \vec{\phi}_x &\equiv \frac{1}{a}\left(\vec{\phi}_x - \vec{\phi}_{x-\hat{\mu}}\right) \ .
\end{aligned}
\tag{2}
$$

The $c_i$ coefficients are fixed by the Symanzik improvement program at one loop [1].

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## 3. ASYMPTOTIC SCALING CORRECTIONS

In order to know the corrections to asymptotic scaling up to four loops we need to compute the perturbative expansion of $\beta^{LAT}$ and $\gamma^{LAT}$ up to four loops for the two considered Symanzik actions. In this proceeding we report the results at three loops while the evaluation for the next order is in progress.

The knowledge of the continuum RGI functions $\beta^{\overline{MS}}$, $\gamma^{\overline{MS}}$ up to four loops allows the calculation of the analogous lattice functions $\beta^{LAT}$, $\gamma^{LAT}$ up to four loops by a three–loop computation of the 2–point 1PI correlation function $\Gamma^{(2)}_{LAT}$.

We have treated the constraint on the norm of the field $\vec{\phi}$ with the standard method [4]. As a consequence the theory is described in terms of a $(n-1)$–component field $\vec{\pi}$ and a measure term has to be added to the action.

The expressions relating the RGI functions on the lattice and in the $\overline{MS}$ scheme are

$$\beta^{LAT}(g) = \frac{Z^g(g)\beta^{\overline{MS}}(g_R)}{1 - g_R \dfrac{\partial Z^g(g)}{\partial g}} \, ,$$

$$\gamma^{LAT}(g) = \gamma^{\overline{MS}}(g_R) - \beta^{LAT}(g)\frac{\partial \log Z^\pi(g)}{\partial g} \, , \qquad (3)$$

where $Z^g$ and $Z^\pi$ are the renormalization constants of the coupling and the field respectively; $g$ and $g_R$ are the bare and the renormalized coupling constants.

We write the perturbative expansions as follows: $\beta^{LAT} = -\beta_0\, g^2 - \beta_1\, g^3 - \beta_2^{LAT}\, g^4 - \ldots$ and $\gamma^{LAT} = \gamma_0\, g + \gamma_1^{LAT}\, g^2 + \ldots$ The coefficients $\beta_0$, $\beta_1$, $\gamma_0$ are universal; our result at three loops for the 0–loop action is

$$\beta^{LAT}_{2,0-loop} = \frac{(n-2)}{16\pi}\Big[(n-2)\Big(-1 + \frac{1}{\pi^2} - 8G_1^S + \frac{5}{6}Y_1 - \frac{7}{48}Y_1^2 - \frac{2}{3}Y_{1,2} + \frac{5}{18}Y_1Y_{1,2}\Big) - 1 + \frac{2}{\pi^2} - \frac{4}{27}G_2^S + \frac{5}{6}Y_1 + \frac{1}{3\pi}Y_1 - \frac{29}{144}Y_1^2 - \frac{1}{18}Y_2 + \frac{5}{216}Y_1Y_2\Big] \, ,$$

$$\gamma^{LAT}_{1,0-loop} = \frac{(n-1)}{24\pi}Y_1 \, ,$$

$$\gamma^{LAT}_{2,0-loop} = \frac{(n-1)}{16\pi}\Big[(n-2)\Big(1 + \frac{1}{\pi^2} + 8G_1^S - \frac{5}{6}Y_1 + \frac{7}{48}Y_1^2 - \frac{2}{3}Y_{1,2} + \frac{5}{18}Y_1Y_{1,2}\Big) + 1 + \frac{4}{27}G_2^S - \frac{5}{6}Y_1 + \frac{37}{144}Y_1^2 + \frac{1}{18}Y_2 - \frac{5}{216}Y_1Y_2\Big] \, , \qquad (4)$$

the notation being

$$Y_{i,j} \equiv \int_{-\pi}^{+\pi} \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{(\Box_q)^i}{(\Pi_q)^j} \, , \qquad Y_i \equiv Y_{i,i} \, ,$$

$$G_1^S \equiv -\int_{-\pi}^{+\pi} \mathcal{D}_3 \, \frac{\sum_\mu \left(\frac{1}{18}\hat{l}_\mu^6 + \frac{1}{144}\hat{l}_\mu^8\right)\Delta^S_{q,k}}{\Pi_q\Pi_k\,(\Pi_l)^2} \, ,$$

$$G_2^S \equiv \int_{-\pi}^{+\pi} \mathcal{D}_3 \, \frac{3\,\hat{l}_\mu^4\,\hat{q}_\mu^4\,\hat{k}_\mu^2 - \frac{1}{4}\,\hat{l}_\mu^4\,\hat{q}_\mu^4\,\hat{k}_\mu^4}{\Pi_q\Pi_k\Pi_l} \, , \qquad (5)$$

where we use the standard notation $\hat{q}_\mu \equiv 2\sin(q_\mu/2)$ and

$$\hat{q}^2 \equiv \sum_\mu \hat{q}_\mu^2 \, , \qquad \Box_q \equiv \sum_\mu \hat{q}_\mu^4 \, ,$$

$$\Pi_q \equiv \hat{q}^2 + \frac{1}{12}\Box_q \, , \quad \Delta^S_{q,k} \equiv \Pi_{q+k} - \Pi_q - \Pi_k \, . \qquad (6)$$

The measure term in the two–loop integrals is

$$\mathcal{D}_3 \equiv \frac{\mathrm{d}^2 q}{(2\pi)^2}\,\frac{\mathrm{d}^2 k}{(2\pi)^2}\,\frac{\mathrm{d}^2 l}{(2\pi)^2}\;(2\pi)^2\;\delta(q+k+l) \, . \qquad (7)$$

Eq.(4) in numerical form is

$$\beta^{LAT}_{2,0-loop} = \frac{(n-2)}{(2\pi)^3}\Big[0.481294 + 0.181889\,(n-2)\Big] \, ,$$

$$\gamma^{LAT}_{1,0-loop} = \frac{1}{(2\pi)^2}\Big[1.07001\,(n-1)\Big] \, ,$$

$$\gamma^{LAT}_{2,0-loop} = \frac{(n-1)}{(2\pi)^3}\Big[2.73365 + 0.355965(n-2)\Big] \qquad (8)$$

The agreement with Ref. [5] is satisfactory within the precision of the numerics. The analogous coefficients for the 1–loop action are a new result,

$$\beta^{LAT}_{2,1-loop} = \beta^{LAT}_{2,0-loop} + \frac{(n-2)}{2\pi}(\eta - 2\zeta) \, ,$$

$$\gamma^{LAT}_{1,1-loop} = \gamma^{LAT}_{1,0-loop} \, ,$$

$$\gamma^{LAT}_{2,1-loop} = \gamma^{LAT}_{2,0-loop} - \frac{(n-1)}{2\pi}\eta - \frac{(n-3)}{\pi}\zeta \, ,$$

$$\eta \equiv (n-1)\left(c_8(\frac{1}{6}Y_1-2)+c_7(\frac{1}{3}Y_1-4)\right.$$
$$\left.+c_9(\frac{2}{3}Y_1-2)\right)+(\frac{4}{3}Y_1-4)(c_7+c_8+\frac{3}{2}c_9)$$
$$+c_6(\frac{5}{2}Y_1+2Y_{1,2}-\frac{1}{6}Y_2-6)+c_5(\frac{4}{3}Y_1+\frac{1}{72}Y_2$$
$$+\frac{5}{144}Y_{2,1}-\frac{1}{864}Y_{3,2}-5)\ ,$$
$$\zeta \equiv (n-1)\left(c_6 Y_{1,2}+c_5(1-\frac{1}{6}Y_1+\frac{1}{144}Y_2)\right) \quad (9)$$

## 4. EFFECTIVE SCHEME

The energy operator $E$ for the two improved actions is (no summation over $\mu$)

$$E=\langle \frac{4}{3}\vec{\phi}_0\cdot\vec{\phi}_{0+\hat{\mu}}-\frac{1}{12}\vec{\phi}_0\cdot\vec{\phi}_{0+2\hat{\mu}}\rangle\ . \quad (10)$$

We write its perturbative expansion as $E=w_0-w_1\ g-w_2\ g^2-\ldots$. An effective scheme [2] is introduced by defining the effective coupling constant $g_E$

$$g_E\equiv\frac{w_0-E^{\rm MC}}{w_1}\ , \quad (11)$$

where $E^{\rm MC}$ is the Monte Carlo measured value of $E$ at the bare coupling $g$. In order to express the asymptotic scaling corrections in terms of $g_E$, we have calculated the perturbative expansion of $E$ up to four loops for the two Symanzik improved actions. We have first put the model into a square box of finite size $L$ with periodic boundary conditions in order to regularize the IR divergences. This procedure has three consequences [6]:
*i*) the momenta are summed, not integrated,
*ii*) the zero modes are absent,
*iii*) a new term, coming from a Faddeev–Popov determinant, has to be added to the action.

The result for $E$ must be finite after the limit $L\to\infty$ has been taken. However partial contributions from individual diagrams can diverge in the thermodynamic limit. These divergent contributions have been algebraically worked out to separate their finite part from the divergent one. At the end all divergences cancel out leaving a result that allows the limit $L\to\infty$. After this limit, the sums over momenta become integrals in the Brillouin zone.

The above–described calculation requires the evaluation of diagrams containing vertices from the Faddeev–Popov determinant [6]. We have checked that the whole contribution of these diagrams up to four loops vanishes in the limit $L\to\infty$.

More details as well as the analytical and the numerical results for the perturbative expansion of $E$ can be found in [3].

## 5. NUMERICAL TECHNICALITIES

We have used three methods to compute the finite lattice integrals in (5), (see [3])
*i*) extrapolation to the infinite lattice size of the result on finite lattices,
*ii*) Gauss method,
*iii*) the coordinate space method [7] extended to the case of improved propagators.
The results of our integrals are

$$\begin{aligned}
Y_1 &= 2.0435764382979844236\ ,\\
Y_2 &= 4.7830710733439886212\ ,\\
Y_{1,2} &= 0.4729502261432961899\ ,\\
Y_{2,1} &= 30.077096804291341057\ ,\\
Y_{3,2} &= 77.324121011413132160\ ,\\
G_1^S &= 0.013948510\ ,\\
G_2^S &= 0.9748227\ . \qquad (12)
\end{aligned}$$

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